运动学

1.1 质点运动学

$$\begin{cases} \text{极坐标：} \vec{r}=\rho\hat{e}_\rho ,\ \vec{v}=\dot{\rho}\hat{e}_\rho+\rho\dot{\varphi}\hat{e}_\varphi ,\ \vec{a}=(\ddot{\rho}-\rho\dot{\varphi}^2)\hat{e}_\rho+(\rho\ddot{\varphi}+2\dot{\rho}\dot{\varphi})\hat{e}_\varphi \\ \text{自然坐标系：} \vec{v}=v\hat{e}_t ,\ \vec{a}=\dot{v}\hat{e}_t+\frac{v^2}{\rho_c}\hat{e}_n \text{ where } \rho_c \text{ is 曲率半径} \end{cases}$$

任意曲线坐标系：由三个相交的曲线族 $C_1 . C_2 . C_3$ 组成，分别对应 $q_1 . q_2 . q_3$ ． 切线方向：$\left|\frac{\partial \vec{r}}{\partial q_i}\right| \hat{e}_{q_i}$

Lami 系数 $H_i=\left|\frac{\partial \vec{r}}{\partial q_i}\right|=\sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2+\left(\frac{\partial y}{\partial q_i}\right)^2+\left(\frac{\partial z}{\partial q_i}\right)^2}$

$$\begin{cases} \vec{v}=\sum_{i=1}^{3}\frac{\partial \vec{r}}{\partial q_i}\cdot\dot{q}_i=\sum_{i=1}^{3}H_i\dot{q}_i\hat{e}_{q_i}=\sum_{i=1}^{3}v_i\hat{e}_{q_i} \\ \vec{a}=\sum_{i=1}^{3}a_i\hat{e}_{q_i} \text{ where } a_i=\frac{1}{H_i}\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i}-\frac{\partial T}{\partial q_i}\right) ,\ T=\frac{1}{2}v^2 \text{ in 正交曲线系} \end{cases}$$

1.2 刚体运动学

微转动满足加法交换律，故角速度为矢量

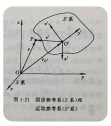
$a=\frac{dv}{dt}=\dot{w}\times r+w\times(w\times r)$

Chasles 定理 自由刚体运动的任一位移可分为 刚体上任一点的平移 + 绕该点的转动

角速度 w 与基点无关，反映了刚体整体运动的特性

转动瞬轴体现为瞬心，瞬心速度为 0，但是加速度不为0

1.3 质点相对运动的运动学



S系为固定参考系，S'系为运动参考系.

$\frac{d}{dt}\vec{e}_x'=\vec{w}\times e_x'$ $\quad\frac{d}{dt}\vec{e}_y'=\vec{w}\times e_y'$ $\quad\frac{d}{dt}\vec{e}_z'=\vec{w}\times e_z'$

$$\Rightarrow \underbrace{\vec{v}}_{\text{绝对}}=\frac{d\vec{r}_0}{dt}+\frac{d\vec{r}'}{dt}=\underbrace{\vec{v}'}_{\text{相对}}+\underbrace{\vec{v}_0+\vec{w}\times\vec{r}'}_{\text{牵连}}$$

$$\underbrace{\vec{a}}_{\text{绝对}}=\underbrace{\vec{a}'}_{\text{相对}}+\underbrace{\vec{a}_0+\frac{d\vec{w}}{dt}\times\vec{r}'+\vec{w}\times(\vec{w}\times\vec{r}')}_{\text{牵连}}+\underbrace{2\vec{w}\times\vec{v}'}_{\text{Coriolis}}$$

Lagrange Equation

2.1 虚功原理

约束：$f(r_i,\dot{r}_i,t)=0$

显含时间 t：非稳定约束， 否则称为 稳定约束。只包含坐标变量的称为 完整约束

体系：k个完整约束，r个非完整约束 ⇒ 独立坐标数为 $3n-k$，自由度为 $3n-k-r$

对于完整系，其中的独立坐标称为广义坐标

虚位移 $\delta\vec{r}=\sum_{\alpha=1}^{s}\frac{\partial \vec{r}}{\partial q_\alpha}\delta q_\alpha$ （含时情况下 $d\vec{r}$ 与 $\delta\vec{r}$ 不同）

$\delta w=\vec{F}_a\cdot\delta\vec{r}+\vec{N}\cdot\delta\vec{r} \xrightarrow{\text{多个质点}} \delta w=\sum_{i=1}^{n}\vec{F}_{ai}\cdot\delta\vec{r}_i+\sum_{i=1}^{n}\vec{N}\cdot\delta\vec{r}_i$

理想约束：$\sum N_i\delta r_i=0$

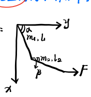
Finally we have：$\sum_{i=1}^{n}\vec{F}_{ai}\cdot\delta\vec{r}_i=0$ 与 Newton 力学平衡方程等价

$\delta w=\sum_{i=1}^{n}F_{ai}\cdot\left(\sum_{\alpha=1}^{s}\frac{\partial r_i}{\partial q_\alpha}\delta q_\alpha\right)=\sum_{\alpha=1}^{s}\left(\sum_{i=1}^{n}F_{ai}\cdot\frac{\partial r_i}{\partial q_\alpha}\right)\delta q_\alpha$

于是 $Q_\alpha=\sum_{i=1}^{n}\vec{F}_{ai}\frac{\partial \vec{r}_i}{\partial q_\alpha}$ 称为广义力的 α 分量

在完整的约束体系中，所有的广义坐标变更都是独立的，故有 $Q_\alpha=0$ $(\alpha=1,2,\cdots,s)$

例：



$w=m_1g\cdot\frac{l_1}{2}\sin\alpha+m_2g\left(l_1\sin\alpha+\frac{l_2}{2}\sin\beta\right)+F\left(l_1\cos\alpha+l_2\cos\beta\right)$

$\frac{\delta w}{\delta\alpha}=\frac{1}{2}m_1gl_1\cos\alpha+m_2gl_1\cos\alpha-Fl_1\sin\alpha=0$

$\frac{\delta w}{\delta\beta}=\frac{1}{2}m_2gl_2\cos\beta-Fl_2\sin\beta=0$

$\Rightarrow \tan\alpha=\frac{\frac{1}{2}m_1+m_2}{F}g \quad \tan\beta=\frac{m_2g}{2F}$

步骤：

(1) 确定体系自由度，选定广义坐标 (2) 受力分析，确定主动力及其作用点，判断理想约束 (3) 虚功

2.2 拉格朗日方程 —— 分析动力学的基本方程

Tips：1. $\frac{\partial \vec{v}_i}{\partial \dot{q}_\alpha}=\frac{\partial \vec{r}_i}{\partial q_\alpha}$

2. $\delta\vec{r}_i=\sum_{\alpha=1}^{s}\frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\delta q_\alpha$

3. $\frac{d}{dt}\frac{\partial \vec{r}_i}{\partial q_\alpha}=\frac{\partial \vec{v}_i}{\partial q_\alpha}$

d'Alembert - Lagrange 方程：$\sum_{i=1}^{n}(F_{ai}-m\ddot{r}_i)\delta r_i=0$

By tips we have：

$\sum_{\alpha=1}^{s}Q_\alpha\delta q_\alpha=\sum_{\alpha=1}^{s}\left\{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha}-\frac{\partial T}{\partial q_\alpha}\right\}\delta q_\alpha$, where $T=\sum_{i=1}^{n}\frac{1}{2}m_iv_i^2$

$\Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha}-\frac{\partial T}{\partial q_\alpha}=Q_\alpha$

保守系的 Lagrange 方程

$\vec{F}_{ai}=-\nabla V_i=-\left(\frac{\partial V_i}{\partial x}\hat{e}_x+\frac{\partial V_i}{\partial y}\hat{e}_y+\frac{\partial V_i}{\partial z}\hat{e}_z\right) \longrightarrow Q_\alpha=-\frac{\partial V}{\partial q_\alpha}$ 其中 $V=\sum_{i=1}^{n}V_i$

Thus we have：

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=0 ,\quad \alpha=1,2,\cdots,s ,\quad L=T-V$$

蕴含着体系的约束、主动力、各质点的运动状态.

2.2.4 广义能量积分，广义动量积分，循环坐标

1. 能量表达式

$\vec{v}_i=\frac{d\vec{r}_i}{dt}=\sum_{\alpha=1}^{s}\frac{\partial \vec{r}}{\partial q_\alpha}\cdot\dot{q}_\alpha+\frac{\partial \vec{r}_i}{\partial t}$, $T=\frac{1}{2}m\vec{v}_i^2$

Then we have $T=T_2+T_1+T_0$, where

$$\begin{cases} T_2=\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}a_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta , & a_{\alpha\beta}=\frac{1}{2}\sum_{i=1}^{n}m_i\frac{\partial \vec{r}}{\partial q_\alpha}\cdot\frac{\partial \vec{r}}{\partial q_\beta} \\ T_1=\sum_{\alpha=1}^{s}b_\alpha\dot{q}_\alpha , & b_\alpha=\sum_{i=1}^{n}\frac{\partial \vec{r}_i}{\partial q_\alpha}\cdot\frac{\partial \vec{r}_i}{\partial t} \\ T_0=c , & c=\frac{1}{2}\sum_{i=1}^{n}m_i\frac{\partial \vec{r}_i}{\partial t}\cdot\frac{\partial \vec{r}_i}{\partial t} \end{cases}$$

特别地，稳定约束中有 $\frac{\partial \vec{r}_i}{\partial t}=0$，从而 $T_1=T_0=0$

(Tips：$\sum_{\alpha=1}^{s}\frac{\partial T_2}{\partial \dot{q}_\alpha}\dot{q}_\alpha=2T_2$，$\sum_{\alpha=1}^{s}\frac{\partial T_1}{\partial \dot{q}_\alpha}\dot{q}_\alpha=T_1$，$\sum_{\alpha=1}^{s}\frac{\partial T_0}{\partial \dot{q}_\alpha}\dot{q}_\alpha=0$)

2. 广义能量积分

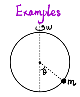
$\frac{d}{dt}(T_2-T_0+V)+\frac{\partial L}{\partial t}=0$ ， 称 T_2-T_0+V 为广义能量 .

当 $\frac{\partial L}{\partial t}=0$ 时，广义能守恒；当 $\frac{\partial \vec{r}}{\partial t}=0$ 时，机械能守恒

3. 广义动量积分与循环坐标

If $\frac{\partial L}{\partial q_\alpha}=0$ then $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}=0$, which means that $\frac{\partial L}{\partial \dot{q}_\alpha}=P_\alpha$ (const)

Examples



讨论 m 的运动

$T=\frac{1}{2}m\left(R^2\dot{\theta}^2+R^2\sin^2\theta\, w^2\right)$ ， $V=-mgR\cos\theta$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}-\frac{\partial L}{\partial \theta}=\frac{d}{dt}(mR^2\dot{\theta})-mw^2R^2\sin\theta\cos\theta+mgR\sin\theta=0$

$\Rightarrow \quad R\ddot{\theta}-(w^2R\cos\theta-g)\sin\theta=0$ 　便为所求的运动微分方程.

Note that：上式对 $d\theta$ 积分可得离心势能下的 conservation of energy.

2.3.1 广义势

采用矢势 $\vec{A}$ 与标势 φ 表示电磁场：$\vec{E}=-\nabla\varphi-\frac{\partial \vec{A}}{\partial t}$ ，$\vec{B}=\vec{\nabla}\times\vec{A}$

引入广义势：$V'=e(\varphi-\vec{v}\cdot\vec{A})$

2.3.2 耗散函数 $f=-b_iv_i$ 问题

$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\alpha}-\frac{\partial T}{\partial q_\alpha}=-\frac{\partial F}{\partial \dot{q}_\alpha}$, where $F=\sum_{i=1}^{n}\frac{1}{2}b_iv_i^2$

2.4 拉格朗日不定乘子法 —— 求约束力

质点被限制在曲面 $f(r,t)$ 上，在理想约束中，有 $\vec{N}=\lambda\nabla f$，λ 称为 Lagrange 不定乘子

设 n 个质点组成的体系中存在 k 个理想完整约束 $f_j(\vec{r},t)=0$ ，$j=1,2,\cdots,k$

和 $s=3n-k$ 个广义坐标 $q_1,\cdots,q_s$.

k 个约束力为：$\vec{N}_j=\lambda_j\vec{\nabla}f_j$

受约束力作用的质点可视为自由质点，广义坐标增加至 3n 个

于是：$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha}-\frac{\partial L}{\partial q_\alpha}=Q_\alpha=\sum_{j=1}^{k}\lambda_j\frac{\partial f_j}{\partial q_\alpha}$

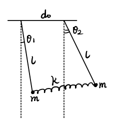
振动

3.1 平衡位置

$dV=\sum_{\alpha=1}^{s}\frac{\partial V}{\partial q_\alpha}dq_\alpha=0$ 且 $d^2V=\sum_{\alpha=1}^{s}\sum_{\beta=1}^{s}\frac{\partial^2 V}{\partial q_\alpha \partial q_\beta}dq_\alpha dq_\beta>0$ ⇒稳定平衡　　将$V(q)$作Taylor展开于平衡位置处

<0 ⇒不稳定平衡

3.2 耦合摆（简正模） Example



微分方程 $\begin{cases} ml^2\ddot\theta_1+\left[ml^2\left(\frac{g}{l}+\frac{k}{m}\right)\theta_1-kl^2\theta_2\right]=0 \\ ml^2\ddot\theta_2+\left[ml^2\left(\frac{g}{l}+\frac{k}{m}\right)\theta_2-kl^2\theta_1\right]=0\end{cases}$ ⇒ $\begin{cases}\ddot\theta_1+w_{gk}^2\theta_1-w_k^2\theta_2=0 \\ \ddot\theta_2+w_{gk}^2\theta_2-w_k^2\theta_1=0\end{cases}$

设 $\theta=A\cos(wt+\varphi)$

于是有 $\begin{vmatrix} w_{gk}^2-w^2 & -w_k^2 \\ -w_k^2 & w_{gk}^2-w^2\end{vmatrix}=0$　从而 $w_1=\sqrt{\frac{g}{l}+\frac{2k}{m}}$　$w_2=\sqrt{\frac{g}{l}}$

记忆已激活

有心运动

4.1 基本运动方程

$L=\frac{1}{2}m\left[\dot r^2+(r\dot\theta)^2+(r\sin\theta\dot\varphi)^2\right]-V(r)$　其中 r、θ、φ 为球坐标下的广义坐标

由 Lagrange 方程：$\begin{cases} m(r\sin\theta)^2\dot\varphi=0 \;\Rightarrow\; \dot\varphi\equiv0 \text{，即有心运动为平面运动} \\ \frac{1}{2}m\left[\dot r^2+(r\dot\theta)^2\right]+V(r)=E_0 \;\text{——}\; m\ddot r-mr\dot\theta^2=F(r) \\ mr^2\dot\theta=J_0 \;\text{——}\; \frac{d}{dt}(mr^2\dot\theta)=0\end{cases}$

4.2 轨道相关

令 $r=\frac{1}{u}$，则 $\dot r=\frac{dr}{d\theta}\cdot\dot\theta=-\frac{J_0}{m}\frac{du}{d\theta}$

同时 $\ddot r=-\left(\frac{J_0}{m}\right)^2u^2\frac{d^2u}{d\theta^2}$

⇒ $\begin{cases} m\left(\frac{J_0}{m}\right)^2u^2\left[\frac{d^2u}{d\theta^2}+u\right]=-F\left(\frac{1}{u}\right) \\ \frac{1}{2}m\left(\frac{J_0}{m}\right)^2\left[\left(\frac{du}{d\theta}\right)^2+u^2\right]+V\left(\frac{1}{u}\right)=E_0\end{cases}$　比内方程（Binet）　一般用这个

平方反比：$V=-ku$

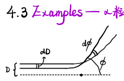
$\Rightarrow\left(\frac{du}{d\theta}\right)^2=\frac{2mE_0}{J_0^2}+\frac{2mk}{J_0^2}u-u^2$　即 $\frac{du}{d\theta}=\pm\sqrt{A^2-\left(u-\frac{mk}{J_0^2}\right)^2}$

从而 $u=\frac{mk}{J_0^2}+\frac{m|k|}{J_0^2}\cdot\sqrt{1+\frac{2E_0J_0^2}{mk^2}}\cos(\theta+\varphi)$

故 $r=\frac{p}{\pm1+e\cos\theta}$，$p=\frac{J_0^2}{m|k|}$，$e=\sqrt{1+\frac{2E_0J_0^2}{mk^2}}$　（$k>0$时取"+"，<0时取"-"）

椭圆时：$a=-\frac{k}{2E_0}$，$b=\frac{J_0}{\sqrt{-2mE_0}}$

4.3 Examples — α粒子散射



由双曲线相关公式：$\cot\frac{\phi}{2}=\frac{4\pi\varepsilon_0 mv^2}{Ze^2}D$

$d\sigma=2\pi D\,dD=-\left(\frac{Ze^2}{4\pi\varepsilon_0 mv^2}\right)^2\frac{\pi\cos\frac{\phi}{2}}{\sin^3\frac{\phi}{2}}d\phi$，$d\Omega=2\pi\sin\phi\, d\phi$

⇒ Rutherford 散射公式：

$$\left|\frac{d\sigma}{d\Omega}\right|=\left(\frac{Ze^2}{8\pi\varepsilon_0 mv^2}\right)^2\Big/\sin^4\frac{\phi}{2}$$

4.4 开普勒方程

$\dot r^2+\left(\frac{J_0}{m}\right)^2\frac{1}{r^2}-\frac{2GM}{r}=\frac{2E_0}{m}$，即 $\frac{r\,dr}{\sqrt{a^2e^2-(r-a)^2}}=\sqrt{\frac{GM}{a^3}}(t-t_0)$

$\psi-e\sin\psi=\sqrt{\frac{GM}{a^3}}(t-t_0)$　where $r=a(1-e\cos\psi)$

$\Rightarrow T^2=\frac{4\pi^2a^3}{GM}$

Rigid Body

$\vec L=\vec w\int r^2dm-\int\vec r(\vec w\cdot\vec r)\,dm$，$L$一般不与$w$平行

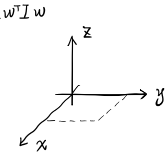
def：$I_x=\int(y^2+z^2)\,dm$，$I_{yz}=I_{zy}=\int yz\,dm$

then：

$$\begin{bmatrix}L_x\\L_y\\L_z\end{bmatrix}=\begin{bmatrix}I_x & -I_{xy} & -I_{xz}\\ -I_{yx} & I_y & -I_{yz}\\ -I_{zx} & -I_{yz} & I_z\end{bmatrix}\begin{bmatrix}w_x\\w_y\\w_z\end{bmatrix}$$

惯量主轴中 $I_{xy}=0$

能量 $T=\frac{1}{2}w^TI\,w$



Hamilton Dynamics

引入新变量 $p_\alpha = \frac{\partial L}{\partial \dot{q}_\alpha}$，广义动量，于是有 $\frac{\partial L}{\partial q_\alpha} = \frac{dp_\alpha}{dt}$

取 p_α 与 q_α $(\alpha = 1, \cdots, s)$ 作为描述体系的独立变量，称为正则变量

Hamilton 函数：$H(q, p, t) = \sum_{\alpha=1}^{s} p_\alpha \dot{q}_\alpha - L(q, \dot{q}, t)$，在一般情况下，Hamilton 函数为广义能量

$$\Rightarrow \quad \dot{q}_\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial q_\alpha}$$

$\frac{d}{dt} H(p, q, t) = \frac{\partial H}{\partial t}$，若 H 不显含 t（即 L 不显含 t），H 为守恒量

Examples

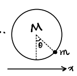
1、r, θ 下质点 m 在有心势场 $V(r)$ 中的运动

$$L = \frac{1}{2} m(\dot{r}^2 + \dot{\theta}^2 r^2) - V(r), \quad p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$H = p_r \cdot \dot{r} + p_\theta \cdot \dot{\theta} - L = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$，不显含 t，故为守恒量

同时 $\frac{\partial H}{\partial \theta} = 0 \Rightarrow p_\theta$ 为守恒量

Then
$$\begin{cases} \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad 即\ p_\theta = p_{\theta_0} \\ \dot{p}_r = -\frac{\partial H}{\partial r} = -\frac{p_{\theta_0}^2}{mr^3} - \frac{dV(r)}{dr} \\ p_\theta = mr^2\dot{\theta} \\ p_r = m\dot{r} \end{cases}$$

2. 

M 的匀质圆盘上固定一个 m，初始：$\theta = 60°$，$\dot{\theta} = 0°$．纯滚动

求 $\theta = 0°$ 时 $\dot{\theta} = ?$

$$L = T - V = \frac{1}{2} \cdot \frac{3}{2} MR^2 \dot{\theta}^2 + mR^2(1 - \cos\theta)\dot{\theta}^2 + mgR\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \frac{3}{2} MR^2 \dot{\theta} + 2mR^2(1 - \cos\theta)\dot{\theta}$$

$H = p_\theta \dot{\theta} - L = \frac{p_\theta^2}{2\left(\frac{3}{2}M + 2(1-\cos\theta)m\right)R^2} - mgR\cos\theta$　　H 守恒可直接解 ✓

$$\begin{cases} \dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{p_\theta^2 m \sin\theta}{\left(\frac{3}{2}M + 2(1-\cos\theta)m\right)^2 R^2} - mgR\sin\theta \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{\left[\frac{3}{2}M + 2(1-\cos\theta)m\right]^2 R^2} \end{cases}$$

没必要用